

Reliable and efficient a posteriori error estimates of DG methods for a frictional contact problem

Fei Wang¹ and Weimin Han²

Abstract. A posteriori error estimators are studied for discontinuous Galerkin methods for solving a frictional contact problem, which is a representative elliptic variational inequality of the second kind. The estimators are derived by relating the error of the variational inequality to that of a linear problem. Reliability and efficiency of the estimators are shown.

Keywords. Elliptic variational inequality, discontinuous Galerkin method, a posteriori error estimators, reliability, efficiency

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1 Introduction

For more than three decades, adaptive finite element method (AFEM) has been an active research field in scientific computing. As an efficient numerical approach, it has been widely used for solving a variety of differential equations. Each loop of AFEM consists of four steps,

Solve \rightarrow Estimate \rightarrow Mark \rightarrow Refine.

That is, in each loop, we first solve the problem on an mesh, then use a posteriori error estimators to mark those elements to be refined, and finally, refine the marked elements and get a new mesh. We can continue this process until the error satisfies certain smallness criterion. The adaptive finite element method can achieve high accuracy with lower memory usage and less computation time.

A posteriori error estimators are computable quantities that indicate the contribution of error on each element to the global error. They are used in adaptive algorithms to indicate which elements need to be refined or coarsened. To capture the true error as precisely as possible, they should have two properties: reliability and efficiency ([1, 4]). Hence, obtaining

¹Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA. School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China. Email: wangfeitwo@163.com

²Department of Mathematics & Program in Applied Mathematical and Computational Sciences, University of Iowa, Iowa City, Iowa 52242, USA.

reliable and efficient error estimators is the key for successful adaptive algorithms. A variety of different a posteriori error estimators have been proposed and analyzed. Many error estimators can be classified as residual type or recovery type ([1, 4]). Various residual quantities are used to capture lost information going from u to u_h , such as residual of the equation, residual from derivative discontinuity and so on. Another type of error estimators is gradient recovery, i.e., $\|G(\nabla u_h) - \nabla u_h\|$ is used to approximate $\|\nabla u - \nabla u_h\|$, where a recovery operator G is applied to the numerical solution u_h to rebuild the gradient of the true solution u . A posteriori error analysis have been well established for standard finite element methods for solving linear partial differential equations, and we refer the reader to [1, 4, 28].

Due to the inequality feature, it is more difficult to develop a posteriori error estimators for variational inequalities (VIs). However, numerous articles can be found on a posteriori error analysis of finite element methods for the obstacle problem, which is an elliptic variational inequality (EVI) of the first kind, e.g., [5, 15, 22, 24, 27, 32]. In [11], Braess demonstrated that a posteriori error estimators for finite element solutions of the obstacle problem can be derived by applying a posteriori error estimates for an associated linear elliptic problem. For VIs of the second kind, in [7, 8, 9, 10], the authors studied a posteriori error estimation and established a framework through the duality theory, but the efficiency was not completely proved. In [29], the ideas in [11] were extended to give a posteriori error analysis for VIs of the second kind. Moreover, a proof was provided for the efficiency of the error estimators.

In recent years, thanks to the flexibility in constructing feasible local shape function spaces and the advantage to capture non-smooth or oscillatory solutions effectively, discontinuous Galerkin (DG) methods have been widely used for solving various types of partial differential equations. When applying h -adaptive algorithm with standard finite element methods, one needs to choose the mesh refinement rule carefully to maintain mesh conformity and shape regularity. In particular, hanging nodes are not allowed without special treatment. For discontinuous Galerkin methods, the approximate functions are allowed to be discontinuous across the element boundaries, so general meshes with hanging nodes and elements of different shapes are accepted. Advantages of DG methods include the flexibility of mesh-refinements and construction of local shape function spaces (hp -adaptivity), and the increase of locality in discretization, which is of particular interest for parallel computing. A historical account of DG methods' development can be found in [16]. In [2, 3], Arnold et al. established a unified error analysis of nine DG methods for elliptic problems and several articles provided a posteriori error analysis of DG methods for elliptic problems (e.g. [6, 12, 14, 21, 23, 25]). Carstensen et al. presented a unified approach to a posteriori error analysis for DG methods in [13]. In [30], the authors extended ideas of the unified framework about DG methods for elliptic problems presented in [3] to solve the obstacle problem and a simplified frictional contact problem, and obtained a priori error estimates, which reach optimal order for linear elements. In [31], reliable a posteriori error estimators of the residual type were derived for DG methods for solving the obstacle problem, and efficiency of the estimators is theoretically explored and numerically confirmed.

A posteriori error analysis of DG methods for the obstacle problem was also studied in [20].

In this paper, we study a posteriori error estimates of DG methods for solving a frictional contact problem. The paper is organized as follows: in Section 2 we introduce a frictional contact problem and the DG schemes for solving it. Then we derive a reliable residual type a posteriori error estimators for the DG methods of a frictional contact problem in Section 3. Finally, we prove efficiency of the proposed error estimators in Section 4.

2 A frictional contact problem and DG formulations

2.1 A frictional contact problem

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open bounded domain with Lipschitz boundary Γ that is divided into two mutually disjoint parts, i.e., $\Gamma = \Gamma_1 \cup \Gamma_2$. Here Γ_1 is a relatively closed subset of Γ , and $\Gamma_2 = \Gamma \setminus \Gamma_1$. Given $f \in L^2(\Omega)$ and a constant $g > 0$, the frictional contact problem is: find $u \in V = H_{\Gamma_1}^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ a.e. on } \Gamma_1\}$ such that

$$a(u, v - u) + j(v) - j(u) \geq (f, v - u) \quad \forall v \in V, \quad (2.1)$$

where (\cdot, \cdot) denotes the L^2 inner product in the domain Ω and

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} u v \, dx, \\ j(v) &= \int_{\Gamma_2} g |v| \, ds. \end{aligned}$$

The frictional contact problem is an example of elliptic variational inequalities of the second kind and has a unique solution $u \in V$ ([18, 19]). Moreover, there exists a unique Lagrange multiplier $\lambda \in L^\infty(\Gamma_2)$ such that

$$a(u, v) + \int_{\Gamma_2} g \lambda v \, ds = (f, v) \quad \forall v \in V, \quad (2.2)$$

$$|\lambda| \leq 1, \quad \lambda u = |u| \quad \text{a.e. on } \Gamma_2. \quad (2.3)$$

From (2.2) and (2.3), we know that the solution u of (2.1) is the weak solution of the following boundary value problem

$$\begin{aligned} -\Delta u + u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_1, \\ \nabla u \cdot \mathbf{n} &= -g\lambda && \text{on } \Gamma_2, \end{aligned}$$

where \mathbf{n} is the unit outward normal vector. For any $v \in V$, set

$$\ell(v) = \int_{\Omega} f v \, dx - \int_{\Gamma_2} g \lambda v \, ds.$$

Then we have by (2.2)

$$a(u, v) = \ell(v) \quad \forall v \in V. \quad (2.4)$$

Similar with the argument in [29], given a triangulation \mathcal{T}_h of Ω , for a Lipschitz subdomain $\omega \subset \Omega$, define

$$a_{\omega,h}(v, w) := \sum_{K \in \mathcal{T}_h} \int_{\omega \cap K} (\nabla v \cdot \nabla w + vw) \, dx$$

and

$$\|v\|_{1,\omega,h} := a_{\omega,h}(v, v)^{1/2}.$$

Then define

$$|\lambda|_{*,\gamma,h} := \sup \left\{ \int_{\gamma} g \lambda v \, ds : v \in H_h^1(\omega), \|v\|_{1,\omega,h} = 1 \right\}, \quad (2.5)$$

where $\gamma \subset \partial\omega \cap \Gamma_2$ is a measurable subset and $H_h^1(\omega) = \{v \in L^2(\omega) : v|_{K \cap \omega} \in H^1(K \cap \omega)\}$. If $\omega = \Omega$ and $\gamma = \Gamma_2$, the subscript ω and γ are omitted. We have

$$|\lambda|_{*,\gamma,h} = \|w\|_{1,\omega,h}, \quad (2.6)$$

where $w \in H_h^1(\omega)$ is the solution of the following auxiliary equation

$$a_{\omega,h}(w, v) = \int_{\gamma} g \lambda v \, ds \quad \forall v \in H_h^1(\omega). \quad (2.7)$$

The formula (2.6) can be proved by an argument similar to that found in [29].

2.2 Discontinuous Galerkin formulations

First, we introduce some notations. Let $\{\mathcal{T}_h\}$ be a family of triangulations of $\overline{\Omega}$ such that the minimal angle condition is satisfied. For a triangulation \mathcal{T}_h , let \mathcal{E}_h be the set of all edges, $\mathcal{E}_h^i \subset \mathcal{E}_h$ the set of all interior edges, $\mathcal{E}_h^b := \mathcal{E}_h \setminus \mathcal{E}_h^i$ the set of all boundary edges, $\mathcal{E}_h^0 \subset \mathcal{E}_h$ the set of all edges not lying on Γ_2 , $\mathcal{E}_h^1 := \mathcal{E}_h^0 \setminus \mathcal{E}_h^i$, $\mathcal{E}_h^2 := \mathcal{E}_h \setminus \mathcal{E}_h^0$, and define $\mathcal{E}(K)$ as the set of sides of K . Let $h_K = \text{diam}(K)$ for $K \in \mathcal{T}_h$, $h_e = \text{length}(e)$ for $e \in \mathcal{E}_h$, and \mathcal{N}_h denote the set of nodes of \mathcal{T}_h . For any element $K \in \mathcal{T}_h$, define the patch set $\omega_K := \cup\{T \in \mathcal{T}_h, T \cap K \neq \emptyset\}$, and for any edge e shared by two elements K^+ and K^- , define $\omega_e := K^+ \cup K^-$. For a scalar-valued function v and a vector-valued function \mathbf{q} , let $v^i = v|_{\partial K^i}$, $\mathbf{q}^i = \mathbf{q}|_{\partial K^i}$, and $\mathbf{n}^i = \mathbf{n}|_{\partial K^i}$ be the

unit normal vector external to ∂K^i with $i = \pm$. Define the average $\{\cdot\}$ and the jump $[\cdot]$ on an interior edge $e \in \mathcal{E}_h^i$ as follows:

$$\begin{aligned}\{v\} &= \frac{1}{2}(v^+ + v^-), & [v] &= v^+ \mathbf{n}^+ + v^- \mathbf{n}^-, \\ \{\mathbf{q}\} &= \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-), & [\mathbf{q}] &= \mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^-.\end{aligned}$$

For a boundary edge $e \in \mathcal{E}_h^b$, we let

$$[v] = v \mathbf{n}, \quad \{q\} = q,$$

where \mathbf{n} is the outward unit normal.

Let us define the following linear finite element spaces

$$\begin{aligned}V_h &= \{v_h \in L^2(\Omega) : v_h|_K \in P_1(K) \ \forall K \in \mathcal{T}_h\}, \\ W_h &= \{\mathbf{w}_h \in [L^2(\Omega)]^2 : \mathbf{w}_h|_K \in [P_1(K)]^2 \ \forall K \in \mathcal{T}_h\}.\end{aligned}$$

We denote by ∇_h the broken gradient whose restriction on each element $K \in \mathcal{T}_h$ is equal to ∇ . Define some seminorms and norms by the following relations:

$$\begin{aligned}\|v\|_K^2 &= \int_K v^2 dx, \quad |v|_{1,K}^2 = \|\nabla v\|_K^2, \quad \|v\|_e^2 = \int_e v^2 ds, \\ \|v\|_{0,h}^2 &= \sum_{K \in \mathcal{T}_h} \|v\|_K^2, \quad |v|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} |v|_{1,K}^2, \quad \|v\|_{1,h}^2 = \|v\|_{0,h}^2 + |v|_{1,h}^2.\end{aligned}$$

Throughout this paper, “ $\lesssim \dots$ ” stands for “ $\leq C \dots$ ”, where C denotes a generic positive constant dependent on the minimal angle condition but not on the element sizes, which may take different values at different occurrences.

Now, let us introduce the Discontinuous Galerkin methods for solving the variational inequality (2.1). Here, we take the local DG method (LDG) as an example to show how to derive a posteriori error estimators of DG methods for solving the frictional contact problem (2.1). The derivation and analysis for the LDG method in this paper can be extended to other DG methods studied in [30]. The LDG method ([17]) for solving the frictional contact problem is to find $u_h \in V_h$ such that

$$B_h(u_h, v_h - u_h) + j(v_h) - j(u_h) \geq (f, v_h - u_h) \quad \forall v_h \in V_h, \quad (2.8)$$

where

$$\begin{aligned}B_h(u, v) &= \int_{\Omega} (\nabla_h u \cdot \nabla_h v + u v) dx - \int_{\mathcal{E}_h^0} [u] \cdot \{\nabla_h v\} ds - \int_{\mathcal{E}_h^0} \{\nabla_h u\} \cdot [v] ds \\ &\quad - \int_{\mathcal{E}_h^i} \beta \cdot [u] [\nabla_h v] ds - \int_{\mathcal{E}_h^i} [\nabla_h u] \beta \cdot [v] ds \\ &\quad + (r_0([u]) + l(\beta \cdot [u]), r_0([v]) + l(\beta \cdot [v])) + \alpha_0^j(u, v).\end{aligned} \quad (2.9)$$

Here $\beta \in [L^2(\mathcal{E}_h^i)]^2$ is a vector-valued function which is constant on each edge of \mathcal{E}_h^i , and $\alpha_0^j(u, v) = \int_{\mathcal{E}_h^0} \eta[u] \cdot [v] ds$ is the penalty term with the penalty weighting function $\eta : \mathcal{E}_h^0 \rightarrow \mathbb{R}$ given by $\eta_e h_e^{-1}$ on each $e \in \mathcal{E}_h^0$, η_e being a positive number on e . For any $\mathbf{w}_h \in W_h$, the lifting operators $r_0 : [L^2(\mathcal{E}_h^0)]^2 \rightarrow W_h$ and $l : L^2(\mathcal{E}_h^i) \rightarrow W_h$ are defined by

$$\int_{\Omega} r_0(\mathbf{q}) \cdot \mathbf{w}_h dx = - \int_{\mathcal{E}_h^0} \mathbf{q} \cdot \{\mathbf{w}_h\} ds, \quad \int_{\Omega} l(v) \cdot \mathbf{w}_h dx = - \int_{\mathcal{E}_h^i} v [\mathbf{w}_h] ds \quad \forall \mathbf{w}_h \in W_h. \quad (2.10)$$

The bilinear form B_h is continuous and elliptic with respect to certain DG-norm, and therefore, in particular, the discrete problem has a unique solution $u_h \in V_h$ (see [3, 30]). Similar to the continuous problem, there exists a unique Lagrange multiplier $\lambda_h \in L^\infty(\Gamma_2)$ such that ([19])

$$B_h(u_h, v_h) + \int_{\Gamma_2} g \lambda_h v_h ds = (f, v_h) \quad \forall v_h \in V_h, \quad (2.11)$$

$$|\lambda_h| \leq 1, \quad \lambda_h u_h = |u_h| \quad \text{a.e. on } \Gamma_2. \quad (2.12)$$

For any $v_h \in V_h$, let

$$\ell_h(v_h) = (f, v_h) - \int_{\Gamma_2} g \lambda_h v_h ds.$$

Then (2.11) becomes

$$B_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_h. \quad (2.13)$$

For any $v \in V$, we know that $[u] = 0$ and $[v] = 0$ on $e \in \mathcal{E}_h^0$. Then we have from (2.2) that

$$B_h(u, v) = a(u, v) = \ell(v) \quad \forall v \in V \quad (2.14)$$

Obviously, u_h is also the finite element approximation of the solution $z \in V$ of the linear problem:

$$B_h(z, v) = \ell_h(v) \quad \forall v \in V, \quad (2.15)$$

which is the weak formulation of the boundary value problem

$$\begin{aligned} -\Delta z + z &= f && \text{in } \Omega, \\ z &= 0 && \text{on } \Gamma_1, \\ \frac{\partial z}{\partial n} &= -g \lambda_h && \text{on } \Gamma_2. \end{aligned} \quad (2.16)$$

2.3 A bridge between $u_h - u$ and $u_h - z$

Next, we relate the error $e := u_h - u$ to $u_h - z$, namely,

$$\|e\|_{1,h} + |\lambda - \lambda_h|_{*,h} \lesssim \|u_h - z\|_{1,h} + \left(\sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|[u_h]\|_e^2 \right)^{1/2}. \quad (2.17)$$

Then we will use this relation to derive a posteriori error estimators for DG solutions of the frictional contact problem by utilizing a posteriori error estimators of the related linear elliptic problem (2.16). Note that a similar approach can be applied to other elliptic variational inequalities of the second kind.

To derive the inequality (2.17), we first define a continuous piecewise linear function in $V_h \cap H_{\Gamma_1}^1(\Omega)$, whose value is close to the numerical solution. For any given $v_h \in V_h$, written $v_h = \sum_{K \in \mathcal{T}_h} \sum_{j=1}^3 \alpha_K^{(j)} \phi_K^{(j)}$, where $\phi_K^{(j)}$, $1 \leq j \leq 3$, are the linear basis functions corresponding to the three vertices of K , we construct a function $\chi \in V_h \cap H_{\Gamma_1}^1(\Omega)$ as follows: At every interior node and the nodes on Γ_2 of the conforming mesh \mathcal{T}_h , the value of χ is set to be the average of the values of v_h computed from all the elements sharing that node, and $\chi = 0$ at the boundary nodes on Γ_1 . For each $\nu \in \mathcal{N}_h$, let $\omega_\nu = \{K \in \mathcal{T}_h : \nu \in K\}$ and denote its cardinality by $|\omega_\nu|$, which is bounded by a constant depending only on the minimal angle condition of the mesh. To each node ν , the associated basis function $\phi^{(\nu)}$ is given by

$$\text{supp} \phi^{(\nu)} = \bigcup_{K \in \omega_\nu} K, \quad \phi^{(\nu)}|_K = \phi_K^{(j)} \text{ for } x_K^{(j)} = \nu.$$

Then we define $\chi \in V_h \cap H_{\Gamma_1}^1(\Omega)$ by

$$\chi = \sum_{\nu \in \mathcal{N}_h} \beta^{(\nu)} \phi^{(\nu)}, \quad \text{where } \beta^{(\nu)} = \frac{1}{|\omega_\nu|} \sum_{x_K^{(j)} = \nu} \alpha_K^{(j)} \quad \text{if } \nu \in \mathcal{N}_h \text{ and } \nu \notin \Gamma_1. \quad (2.18)$$

For nonconforming meshes, let \mathcal{N}_h^0 be the set of all hanging nodes. Then we construct χ from v_h same as conforming mesh case on all the nodes $\nu \in \mathcal{N}_h \setminus \mathcal{N}_h^0$. For an upper bound of the error $v_h - \chi$, we quote a result from [21] (which is Theorem 2.2 there for conforming meshes; the same result also holds for nonconforming meshes, which is Theorem 2.3 in [21]).

Lemma 2.1 *Let \mathcal{T}_h be a conforming triangulation. Then for any $v_h \in V_h$, we can construct a continuous function $\chi \in V_h \cap H_{\Gamma_1}^1(\Omega)$ from v_h , such that*

$$\sum_{K \in \mathcal{T}_h} \|v_h - \chi\|_{i,K}^2 \leq C \sum_{e \in \mathcal{E}_h^0} h_e^{1-2i} \|[v_h]\|_e^2, \quad i = 0, 1, \quad (2.19)$$

where the constant C is independent of mesh size and v_h but which may depend on the lower bound of the minimal angle of the elements in \mathcal{T}_h .

Now, let us derive the inequality (2.17). From (2.14) and (2.15), for all $v \in V$, we have

$$B_h(u_h - u, v) = B_h(u_h - z, v) + B_h(z - u, v) = B_h(u_h - z, v) + \int_{\Gamma_2} g(\lambda - \lambda_h) v \, ds.$$

By the definition (2.9) and noticing $[v] = 0$ on each $e \in \mathcal{E}_h^0$, the above equation becomes

$$\begin{aligned} \tilde{a}(e, v) &- \int_{\mathcal{E}_h^0} [e] \cdot \{\nabla_h v\} ds - \int_{\mathcal{E}_h^i} \beta \cdot [e] [\nabla_h v] ds \\ &= \tilde{a}(u_h - z, v) - \int_{\mathcal{E}_h^0} [u_h - z] \cdot \{\nabla_h v\} ds \\ &\quad - \int_{\mathcal{E}_h^i} \beta \cdot [u_h - z] [\nabla_h v] ds + \int_{\Gamma_2} g(\lambda - \lambda_h) v ds, \end{aligned}$$

where

$$\tilde{a}(u, v) = \int_{\Omega} (\nabla_h u \cdot \nabla_h v + u v) dx.$$

Then,

$$\tilde{a}(e, v) = \tilde{a}(u_h - z, v) - \int_{\mathcal{E}_h^0} [u - z] \cdot \{\nabla_h v\} ds - \int_{\mathcal{E}_h^i} \beta \cdot [u - z] [\nabla_h v] ds + \int_{\Gamma_2} g(\lambda - \lambda_h) v ds.$$

Note that $[u - z] = 0$ on each $e \in \mathcal{E}_h^0$. We have

$$\tilde{a}(e, v) = \tilde{a}(u_h - z, v) + \int_{\Gamma_2} g(\lambda - \lambda_h) v ds. \quad (2.20)$$

Let $\chi \in V_h \cap H_{\Gamma_1}^1(\Omega)$ be the function constructed from u_h , satisfying (2.19) for $v_h = u_h$. Taking $v := \chi - u = \chi - u_h + u_h - u$ in (2.20) and using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|e\|_{1,h}^2 &\leq \|u_h - z\|_{1,h} (\|\chi - u_h\|_{1,h} + \|e\|_{1,h}) + \|e\|_{1,h} \|\chi - u_h\|_{1,h} + \int_{\Gamma_2} g(\lambda - \lambda_h) (\chi - u) ds \\ &= \|e\|_{1,h} (\|u_h - z\|_{1,h} + \|\chi - u_h\|_{1,h}) + \|u_h - z\|_{1,h} \|\chi - u_h\|_{1,h} \\ &\quad + \int_{\Gamma_2} g(\lambda - \lambda_h) (\chi - u) ds \\ &\leq \frac{1}{2} \|e\|_{1,h}^2 + \frac{1}{2} (\|u_h - z\|_{1,h} + \|\chi - u_h\|_{1,h})^2 + \|u_h - z\|_{1,h} \|\chi - u_h\|_{1,h} \\ &\quad + \int_{\Gamma_2} g(\lambda - \lambda_h) (\chi - u) ds. \end{aligned} \quad (2.21)$$

Note that by (2.3) and (2.12), we have

$$\begin{aligned} \int_{\Gamma_2} g(\lambda - \lambda_h) (u_h - u) ds &= \int_{\Gamma_2} g \lambda u_h ds - \int_{\Gamma_2} g \lambda u ds - \int_{\Gamma_2} g \lambda_h u_h ds + \int_{\Gamma_2} g \lambda_h u ds \\ &\leq \int_{\Gamma_2} g |u_h| ds - \int_{\Gamma_2} g |u| ds - \int_{\Gamma_2} g |u_h| ds + \int_{\Gamma_2} g |u| ds = 0. \end{aligned}$$

and

$$\begin{aligned} \int_{\Gamma_2} g(\lambda - \lambda_h) (\chi - u_h) ds &\leq |\lambda - \lambda_h|_{*,h} \|\chi - u_h\|_{1,h} \\ &\leq \epsilon |\lambda - \lambda_h|_{*,h}^2 + \frac{1}{4\epsilon} \|\chi - u_h\|_{1,h}^2. \end{aligned}$$

Hence,

$$\|e\|_{1,h}^2 \lesssim \|u_h - z\|_{1,h}^2 + \|\chi - u_h\|_{1,h}^2 + \epsilon |\lambda - \lambda_h|_{*,h}^2. \quad (2.22)$$

Recalling (2.6), we have

$$|\lambda - \lambda_h|_{*,h} = \|u - z\|_{1,h} \leq \|e\|_{1,h} + \|u_h - z\|_{1,h}.$$

Then, we obtain the following result

$$\|e\|_{1,h} + |\lambda - \lambda_h|_{*,h} \lesssim \|u_h - z\|_{1,h} + \|\chi - u_h\|_{1,h}.$$

Using (2.19) to bound $\|\chi - u_h\|_{1,h}$, the above inequality can be rewritten as

$$\|e\|_{1,h} + |\lambda - \lambda_h|_{*,h} \lesssim \|u_h - z\|_{1,h} + \left(\sum_{e \in \mathcal{E}_h^0} h_e^{-1} \| [u_h] \|_e^2 \right)^{1/2}. \quad (2.23)$$

The relation (2.23) serves as a starting point for derivation of reliable and efficient error estimators of DG methods for a frictional contact problem. In this paper, we focus on the derivation and analysis of residual type error estimators derived from the inequality (2.23). A similar approach can also be applied to recovery type error estimators.

3 Reliable residual-type estimators

Now we follow the ideas in [29] to obtain a posteriori error estimators of DG methods for solving the frictional contact problem. The detailed derivation and analysis of a posteriori error estimators is given for the LDG method [17]. For other DG methods discussed in [30], similar results could be obtained by similar arguments.

To bound the first term $\|u_h - z\|_{1,h}$, we recall one result in [13]. Note that the a posteriori error analysis in [13] was only for the Poisson problem with homogenous Dirichlet boundary condition, but it is easy to extend the result to general elliptic problems with Neumann boundary conditions. For the second-order elliptic problem

$$-\Delta u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_1, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma_2,$$

rewrite it as the first order system

$$p = \nabla u, \quad -\nabla \cdot p + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_1, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma_2. \quad (3.1)$$

Then the DG formulation for this problem is

$$\int_{\Omega} p_h \cdot \tau_h dx = - \int_{\Omega} u_h \nabla_h \cdot \tau_h dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \hat{u}_h n_K \cdot \tau_h ds \quad \forall \tau_h \in W_h, \quad (3.2)$$

$$\int_{\Omega} (p_h \cdot \nabla_h v_h + u_h v_h) dx = \int_{\Omega} f v_h dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \hat{p}_h \cdot n_K v_h ds \quad \forall v_h \in V_h, \quad (3.3)$$

where \hat{u}_h and \hat{p}_h are numerical fluxes. Different choices of the numerical fluxes lead to different DG methods. The following theorem (see [13]) holds for the LDG method and other methods discussed in [3].

Theorem 3.1 *Assume $u \in H_{\Gamma_1}^1(\Omega)$ and $p \in W := [L^2(\Omega)]^2$ are the solution of the problem (3.1), and $u_h \in V_h$ and $p_h \in W_h$ are the solution of the problem (3.2)–(3.3). Then,*

$$\|p - p_h\| \leq C(\eta_* + \zeta_*),$$

where

$$\begin{aligned} \eta_*^2 &:= \sum_{K \in \mathcal{T}_h} h_K^2 \|\operatorname{div} p_h - u_h + f\|_K^2 + \sum_{e \in \mathcal{E}_h^i} h_e \|[p_h]\|_e^2 + \sum_{e \in \mathcal{E}_h^2} h_e \|p_h \cdot n - g\|_e^2, \\ \zeta_*^2 &:= \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|[u_h]\|_e^2 \end{aligned}$$

and C is a mesh-size independent constant which depends only on the domain Ω and the minimal angle condition.

From the relation between p_h and u_h ([3, 13]), we deduce the following result.

Corollary 3.2 *With the same notation as in Theorem 3.1, we have*

$$\|\nabla u - \nabla_h u_h\| \leq C(\eta + \zeta_*),$$

where

$$\eta^2 := \sum_{K \in \mathcal{T}_h} h_K^2 \|\Delta u_h - u_h + f\|_K^2 + \sum_{e \in \mathcal{E}_h^i} h_e \|[\nabla_h u_h]\|_e^2 + \sum_{e \in \mathcal{E}_h^2} h_e \|\nabla_h u_h \cdot n - g\|_e^2.$$

Proof. By [26, Lemma 7.2],

$$\|r_0([v_h])\|^2 \leq C \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|[v_h]\|_e^2, \quad \|l(\beta \cdot [v_h])\|^2 \leq C \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|[v_h]\|_e^2, \quad \forall v_h \in V_h.$$

From [3, (3.9)], we know that

$$p_h = \nabla_h u_h - r_0([\hat{u}_h - u_h]) - l(\{\hat{u}_h - u_h\}).$$

Then

$$\begin{aligned} \|\nabla u - \nabla_h u_h\| &\leq \|\nabla u - p_h\| + \|p_h - \nabla_h u_h\| \\ &\leq C(\eta_* + \zeta_*) + \|r_0([\hat{u}_h - u_h])\| + \|l(\{\hat{u}_h - u_h\})\|. \end{aligned}$$

From the choices of numerical fluxes \hat{u}_h in Table 3.1 of [3], we have

$$[\hat{u}_h - u_h] = -[u_h] \text{ or } 0, \quad \{\hat{u}_h - u_h\} = -\beta \cdot [u_h] \text{ or } 0.$$

So

$$\|r_0([\hat{u}_h - u_h])\| \leq C \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|[u_h]\|_e^2, \quad \|l(\{\hat{u}_h - u_h\})\| \leq C \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|[u_h]\|_e^2,$$

which implies

$$\|p_h - \nabla_h u_h\| \leq \zeta_* \quad \text{and} \quad \|\nabla u - \nabla_h u_h\| \leq C(\eta_* + \zeta_*).$$

Finally, by the inverse inequality and trace inequality, we get

$$\begin{aligned} \eta_*^2 &= \sum_{K \in \mathcal{T}_h} h_K^2 \|\operatorname{div} p_h - u_h + f\|_K^2 + \sum_{e \in \mathcal{E}_h^i} h_e \|[p_h]\|_e^2 + \sum_{e \in \mathcal{E}_h^2} h_e \|p_h \cdot n - g\|_e^2 \\ &\leq 2 \left(\eta^2 + \sum_{K \in \mathcal{T}_h} h_K^2 \|\operatorname{div}(p_h - \nabla_h u_h)\|_K^2 + \sum_{e \in \mathcal{E}_h^i} h_e \|p_h - \nabla_h u_h\|_e^2 + \sum_{e \in \mathcal{E}_h^2} h_e \|(p_h - \nabla_h u_h) \cdot n\|_e^2 \right) \\ &\leq 2\eta^2 + 2 \sum_{K \in \mathcal{T}_h} h_K^2 \|\operatorname{div}(p_h - \nabla_h u_h)\|_K^2 + C \left(\sum_{K \in \mathcal{T}_h} \|p_h - \nabla_h u_h\|_K^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |p_h - \nabla_h u_h|_{1,K}^2 \right) \\ &\leq 2\eta^2 + C \sum_{K \in \mathcal{T}_h} \|p_h - \nabla_h u_h\|_K^2 = 2\eta^2 + C \|p_h - \nabla_h u_h\|^2 \leq 2\eta^2 + C\zeta_*^2. \end{aligned}$$

Therefore, $\eta_* \leq C(\eta + \zeta_*)$ and the result is proved. \blacksquare

Define the interior residuals and edge-based jumps

$$\begin{aligned} R_K &:= \Delta u_h - u_h + f \quad \text{for each } K \in \mathcal{T}_h, \\ R_e &:= [\nabla_h u_h] \quad \text{for each } e \in \mathcal{E}_h^i, \quad R_e := \nabla_h u_h \cdot n + g\lambda_h \quad \text{for each } e \in \mathcal{E}_h^2. \end{aligned}$$

Then the local estimators are

$$\eta_K := \left(h_K^2 \|R_K\|_K^2 + \frac{1}{2} \sum_{e \in \partial K \cap \mathcal{E}_h^i} h_e \|R_e\|_e^2 + \sum_{e \in \partial K \cap \mathcal{E}_h^2} h_e \|R_e\|_e^2 \right)^{1/2}, \quad (3.4)$$

$$\eta_{\partial K} := \left(\frac{1}{2} \sum_{e \in \partial K \cap \mathcal{E}_h^i} h_e^{-1} \|[u_h]\|_e^2 + \sum_{e \in \partial K \cap \mathcal{E}_h^1} h_e^{-1} \|[u_h]\|_e^2 \right)^{1/2}. \quad (3.5)$$

Applying Corollary 3.2 to $\|u_h - z\|_{1,h}$, we obtain from (2.17)

$$\|e\|_{1,h} + |\lambda - \lambda_h|_{*,h} \lesssim \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 + \sum_{K \in \mathcal{T}_h} \eta_{\partial K}^2 \right)^{1/2}. \quad (3.6)$$

Theorem 3.3 *Let $u \in H^2(\Omega)$ and u_h solve (2.1) and (2.8) respectively. Then we have the bound (3.6).*

4 Efficiency of the estimators

Now we consider lower bounds of the estimators. We follow the standard argument of lower bounds of residual error estimators for elliptic problems, see [1, pp. 28–31]. First, we introduce the bubble functions. Let $K \in \mathcal{T}_h$, and let λ_1, λ_2 and λ_3 be the barycentric coordinates on K . Then the interior bubble function φ_K is defined by

$$\varphi_K = 27\lambda_1\lambda_2\lambda_3$$

and the three edge bubble functions are given by

$$\tau_1 = 4\lambda_2\lambda_3, \quad \tau_2 = 4\lambda_1\lambda_3, \quad \tau_3 = 4\lambda_1\lambda_2.$$

We list properties of bubble functions stated in Theorems 2.2 and 2.3 of [1] in the form of a lemma.

Lemma 4.1 *For each $K \in \mathcal{T}_h$, $e \subset \partial K$, let φ_K and τ_e be the corresponding interior and edge bubble functions. Let $P(K) \subset H^1(K)$ and $P(e) \subset H^1(e)$ be finite-dimensional spaces of functions defined on K or e . Then there exists a constant C independent of h_K such that for all $v \in P(K)$,*

$$C^{-1}\|v\|_K^2 \leq \int_K \varphi_K v^2 dx \leq C\|v\|_K^2, \quad (4.1)$$

$$C^{-1}\|v\|_K \leq \|\varphi_K v\|_K + h_K |\varphi_K v|_{1,K} \leq C\|v\|_K, \quad (4.2)$$

$$C^{-1}\|v\|_e^2 \leq \int_e \tau_e v^2 ds \leq C\|v\|_e^2, \quad (4.3)$$

$$h_K^{-1/2}\|\tau_e v\|_K + h_K^{1/2}|\tau_e v|_{1,K} \leq C\|v\|_e. \quad (4.4)$$

Denote

$$a_K(u, v) = \int_K (\nabla u \cdot \nabla v + uv) dx.$$

Then for $u, v \in H^1(\Omega)$,

$$a(u, v) = \sum_{K \in \mathcal{T}_h} a_K(u, v).$$

For all $v \in H_{\Gamma_1}^1(\Omega)$, noting that $[v] = 0$ and $[u - z] = 0$ on $e \in \mathcal{E}_h^0$, we have

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} a_K(e, v) &= \sum_{K \in \mathcal{T}_h} a_K(u_h - z, v) + a(z - u, v) = \sum_{K \in \mathcal{T}_h} a_K(u_h - z, v) + B_h(z - u, v) \\ &= \sum_{K \in \mathcal{T}_h} \int_K (\nabla(u_h - z) \cdot \nabla v + (u_h - z)v) dx + \int_{\Gamma_2} g(\lambda - \lambda_h)v ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K (-\Delta(u_h - z) + u_h - z)v dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \nabla(u_h - z) \cdot n_K v ds \\ &\quad + \int_{\Gamma_2} g(\lambda - \lambda_h)v ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K (-\Delta u_h + u_h - f)v dx + \sum_{e \in \mathcal{E}_h^i} \int_e [\nabla u_h] \cdot v ds \\ &\quad + \sum_{e \in \mathcal{E}_h^2} \int_e (\nabla u_h \cdot n + g\lambda_h)v ds + \int_{\mathcal{E}_h^2} g(\lambda - \lambda_h)v ds. \end{aligned} \tag{4.5}$$

For each $K \in \mathcal{T}_h$, φ_K and τ_e are respectively the interior and edge bubble functions on K or $e \in \mathcal{E}_h^i \cup \mathcal{E}_h^2$. \bar{R}_K is an approximation to the interior residual R_K from a suitable finite-dimensional subspace. In (4.5), choose $v = \bar{R}_K \varphi_K$ on element K . We know φ_K vanishes on the boundary of K by its definition, so v can be extended to be zero on the rest of domain as a continuous function. Therefore, we get

$$a_K(e, \bar{R}_K \varphi_K) = \int_K R_K \bar{R}_K \varphi_K dx.$$

Then

$$\int_K \bar{R}_K^2 \varphi_K dx = \int_K \bar{R}_K (\bar{R}_K - R_K) \varphi_K dx + a_K(e, \bar{R}_K \varphi_K).$$

Applying the Cauchy-Schwarz inequality and Lemma 4.1, we obtain

$$\begin{aligned} \int_K \bar{R}_K (\bar{R}_K - R_K) \varphi_K dx &\leq \|\bar{R}_K \varphi_K\|_K \|\bar{R}_K - R_K\|_K \lesssim \|\bar{R}_K\|_K \|\bar{R}_K - R_K\|_K, \\ a_K(e, \bar{R}_K \varphi_K) &\leq \|e\|_{1,K} \|\bar{R}_K \varphi_K\|_{1,K} \lesssim h_K^{-1} \|e\|_{1,K} \|\bar{R}_K\|_K. \end{aligned}$$

Use Lemma 4.1 again,

$$\|\bar{R}_K\|_K^2 \lesssim \int_K \bar{R}_K^2 \varphi_K dx.$$

Combining the above relations, we obtain

$$\|\bar{R}_K\|_K \lesssim \|\bar{R}_K - R_K\|_K + h_K^{-1} \|e\|_{1,K}.$$

Finally, by the triangle inequality $\|R_K\|_K \leq \|R_K - \bar{R}_K\|_K + \|\bar{R}_K\|_K$, we get

$$\|R_K\|_K \lesssim \|\bar{R}_K - R_K\|_K + h_K^{-1} \|e\|_{1,K}.$$

Now choose the finite-dimensional subspace from which the \bar{R}_K come as the function space spanned by the local nodal basis $\phi_K^{(i)}$ with $i = 1, 2, 3$. Then, $\|\bar{R}_K - R_K\|_K$ reduces to $\|f - \bar{f}\|_K$ where we take

$$\bar{f} = \sum_{i=1}^3 f^i \phi_K^{(i)} \quad \text{with} \quad f^i = (f, \phi_K^{(i)})_K / (1, \phi_K^{(i)})_K. \quad (4.6)$$

For $e \in \mathcal{E}_h^2$, we obtain

$$a_{\omega_e}(u_h - u, \bar{R}_e \tau_e) = \int_{\omega_e} R_K \bar{R}_e \tau_e dx + \int_e R_e \bar{R}_e \tau_e ds + \int_e g(\lambda - \lambda_h) \bar{R}_e \tau_e ds$$

and therefore

$$\begin{aligned} \int_e \bar{R}_e^2 \tau_e ds &= \int_e \bar{R}_e (\bar{R}_e - R_e) \tau_e ds + a_{\omega_e}(u_h - u, \bar{R}_e \tau_e) \\ &\quad - \int_{\omega_e} R_K \bar{R}_e \tau_e dx - \int_e g(\lambda - \lambda_h) \bar{R}_e \tau_e ds. \end{aligned}$$

From Lemma 4.1, we estimate the terms in above relation as

$$\begin{aligned} C^{-1} \|\bar{R}_e\|_e^2 &\leq \int_e \bar{R}_e^2 \tau_e ds, \\ \int_e \bar{R}_e (\bar{R}_e - R_e) \tau_e ds &\leq \|\bar{R}_e \tau_e\|_e \|\bar{R}_e - R_e\|_e \leq C \|\bar{R}_e\|_e \|\bar{R}_e - R_e\|_e, \\ a_{\omega_e}(u_h - u, \bar{R}_e \tau_e) &\leq \|u_h - u\|_{1,\omega_e} \|\bar{R}_e \tau_e\|_{1,\omega_e} \leq C h_e^{-1/2} \|u_h - u\|_{1,\omega_e} \|\bar{R}_e\|_e, \\ \int_{\omega_e} R_K \bar{R}_e \tau_e dx &\leq \|R_K\|_{\omega_e} \|\bar{R}_e \tau_e\|_{\omega_e} \leq C h_e^{1/2} \|R_K\|_{\omega_e} \|\bar{R}_e\|_e, \\ \int_e g(\lambda - \lambda_h) \bar{R}_e \tau_e ds &\leq |\lambda - \lambda_h|_{*,e} \|\bar{R}_e \tau_e\|_{1,\omega_e} \leq C h_e^{-1/2} |\lambda - \lambda_h|_{*,e} \|\bar{R}_e\|_e. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \|R_e\|_e &\leq \|\bar{R}_e\|_e + \|R_e - \bar{R}_e\|_e \\ &\leq C (h_e^{-1/2} \|u_h - u\|_{1,\omega_e} + h_e^{-1/2} |\lambda - \lambda_h|_{*,e} + h_e^{1/2} \|R_K - \bar{R}_K\|_{\omega_e} + \|R_e - \bar{R}_e\|_e). \end{aligned} \quad (4.7)$$

For $e \in \mathcal{E}_h^i$, let \bar{R}_e be an approximation to the jump R_e from a suitable finite-dimensional space and let $v = \bar{R}_e \tau_e$ in (4.5). By a similar argument, we have

$$\|R_e\|_e \leq C (h_e^{-1/2} \|u_h - u\|_{1,\omega_e} + h_e^{1/2} \|R_K - \bar{R}_K\|_{\omega_e} + \|R_e - \bar{R}_e\|_e).$$

Note that $\Delta u_h + u_h$ in K and $\partial u_h / \partial n_e$ on e are polynomials. Hence, the terms $\|R_K - \bar{R}_K\|_K$ and $\|R_e - \bar{R}_e\|_e$ can be replaced by $\|f - \bar{f}\|_K$ and $\|\lambda_h - \bar{\lambda}_h\|_e$, with discontinuous piecewise polynomial approximations $\bar{\lambda}_h$. Then we obtain the efficiency bound of the local error indicator η_K .

Theorem 4.2 *Let u and u_h be the solutions of (2.1) and (2.8), respectively, and η_K be the estimator (3.4). Then*

$$\eta_K \leq C \left(|u - u_h|_{\omega_K} + \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_2} |\lambda - \lambda_h|_{*,e} + h_K \|f - f_h\|_{\omega_K} + \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_2} h_e \|\lambda_h - \bar{\lambda}_h\|_e^2 \right), \quad (4.8)$$

where the constant C is dependent on the angle condition and independent of h_K .

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